Math. Meth. in the Appl. Sci. 5 (1983) 84 – 96
AMS subject classification: 35 R 35, 35 K 05

A Critical Case for the Solvability of Stefan-like Problems*)

A. Fasano and M. Primicerio, Firenze

Communicated by K. P. Hadeler

We consider a one-phase one-dimensional Stefan problem with general data with the aim to investigate some open questions on existence of classical solutions. We show how existence and nonexistence are discriminated by the behavior of the initial datum in the neighborhood of the starting point of the free boundary.

1 Introduction

An extensive literature has developed on the following free boundary problem: find $\{T, s, z\}$ such that

(1.1) $z_{xx} - z_t = 0\,, \quad \text{in} \quad D_T \equiv \{(x,t): 0 < x < s(t),\, 0 < t < T\}$

(1.2) $s(0) = b > 0\,,$

(1.3) $z(x,0) = h(x)\,, \qquad 0 < x < b\,,$

(1.4) $z(0,t) = f(t)\,, \qquad 0 < t < T\,,$

(1.5) $z(s(t),t) = 0\,, \qquad 0 < t < T\,,$

(1.6) $z_x(s(t),t) = -\dot{s}(t)\,, \qquad 0 < t < T\,.$

We refer to [1], Part I, for the definition of a classical solution to (1.1) – (1.6). We assume that the data h, f are bounded and piecewise continuous.

It is well known that if h, f are nonnegative then a unique classical solution exists, irrespective of the value of $\limsup_{x \to b^-} h(x)$; moreover, $s(t)$ decreases to b as $t \to 0$ not slower than $t^{1/2}$ (cf. [2]).

Existence and uniqueness of classical solutions when the sign of data is not specified have been proved in [1], Part II, under the assumption $|h(x)| \leqslant \text{const}\,|x - b|^{\beta}$ for any $\beta > 0$.

*) Work performed under the auspices of the Italian C.N.R., G.N.F.M.

In [3] a special case with $h(b) < 0$ is considered: namely, a constant $h_0 > -1$ is found such that if $h(b) > h_0$, then the problem has a solution (Thm. 2).

On the other hand, in [4], where (1.4) is replaced by $z_x(0,t) = 0$, we proved that no solution can exist if $h(x) \leqslant -1$ in $(0,b)$.

The aim of the present paper is to show how the solvability of (1.1) – (1.6) is critically related to the behavior of $h(x)$ in a neighborhood of $x = b$, giving a nearly complete answer to the question of well-posedness of (1.1) – (1.6) in the classical sense.

For a better interpretation of the results, the following alternative formulation of the problem can be useful. Define

$$(1.7) \quad g(x) \equiv \int_x^b d\xi \int_\xi^b dy(h(y) + 1)$$

and note that the well known transformation

$$(1.8) \quad u(x,t) = \int_x^{s(t)} d\xi \int_\xi^{s(t)} dy(z(y,t) + 1), \qquad 0 \leqslant x \leqslant s(t),$$

leads from a solution $\{T, s, z\}$ of (1.1) – (1.6) to a solution $\{T, s, u\}$ of the following problem

$$(1.9) \quad u_{xx} - u_t = 1\,, \quad \text{in} \quad D_T,$$

$$(1.10) \quad s(0) = b\,,$$

$$(1.11) \quad u(x,0) = g(x)\,, \qquad 0 < x < b\,,$$

$$(1.12) \quad u(0,t) = g(0) + \int_0^t f(\tau)d\tau\,; \qquad 0 < t < T\,,$$

$$(1.13) \quad u(s(t),t) = 0\,, \qquad 0 < t < T\,,$$

$$(1.14) \quad u_x(s(t),t) = 0\,, \qquad 0 < t < T\,.$$

Problem (1.9) – (1.14) can be thought as the mathematical model of the diffusion of a substance in an absorbing medium: in this case $u(x,t)$ represents the concentration, and the free boundary $x = s(t)$ delimitates the region where the substance is present. In particular this problem has been considered (cf. [10]) in the study of the diffusion and consumption of oxygen in a living tissue.

In the spirit of problem (1.9) – (1.14), it is conceivable that an important role in discriminating well-posed and ill-posed problems should be played by the initial concentration $g(x)$.

A first result is the following

Theorem 1.1 (non-existence) *Assume that for some $\sigma > 0$ the following condition is satisfied,*

$$g(x) \leqslant 0\,, \quad \textit{for} \quad x \in (b - \sigma, b)\,.$$

Then problem (1.1) – (1.6) *has no solution.*

The proof will be given in Section 2.

Because of the transformation (1.8), the theorem is simply related to the fact that problem (1.9) – (1.14) only makes sense if the initial concentration is strictly positive. Actually, the condition imposed on the initial datum (here, as in the two following theorems) is restricted to hold in a left hand neighborhood of $x = b$, because we are interested in more general results and our well-posedness is always intended in a "local" sense.

In view of the above reasoning, it can be conjectured that the condition

$$\text{(H)} \qquad \int_x^b \int_\xi^b (h(y) + 1)\,\mathrm{d}y\,\mathrm{d}\xi > 0$$

for x in some left neighborhood of b is necessary and sufficient for the well posedness of (1.1) – (1.6) in a classical sense*).

Actually, the existence and uniqueness theorems which we prove in this paper will require slightly stronger assumptions; but, roughly speaking, they mean that – as far as problem of diffusion-consumption is considered – the mathematical problem has a unique classical solution when it is related (at least in a neighborhood of the initial location of the free boundary) to a physically meaningful situation: i.e. when the initial concentration of the diffusing substance is positive.

Concerning uniqueness we have the following theorem.

Theorem 1.2 (uniqueness) *Assume*

(H1) *there exists* $\sigma > 0$ *such that* $h(x) \geqslant -1, h(x) \not\equiv -1$ *in* $(b - \sigma, b)$.

Then problem (1.1) – (1.6) *has at most one solution.*

The proof is found in Section 2.

Section 3 to 6 are devoted to the proof of the following theorem.

Theorem 1.3 (existence) *Assume*

(H2) *there exists* $\sigma > 0$ *such that* $h(x) > -1$ *in* $(b - \sigma, b)$.

Then problem (1.1) – (1.6) *possesses a (unique) classical solution.*

The sketch of the proof is as follows: in Sec. 3 we prove a monotone dependence lemma; in Sec. 4 we prove existence in the special case

$$\text{(H3)} \qquad \liminf_{x \to b^-} h(x) > -1\,;$$

these results will be used in Sec. 5 to get existence for a model problem in which $\lim_{x \to b^-} h(x) = -1$, while Sec. 6 is the final approach to the general case: the main

*) In [5], where problem (1.9) – (1.14) is considered with (1.12) replaced by $u_x(0,t) = 0$, condition (H) is assumed to hold for all $x \in (0,b)$ (together with additional requirements such as $h \leqslant 0$) for proving existence in a weaker sense.

tool will be the construction of approximating solutions solving approximating free boundary problems, using the results of Sec. 5 to provide barriers as solutions of model problems. In the last part an important role is played by the convexity of the free boundary in the model problem.

2 Proof of Theorems 1.1 and 1.2

(I) Proof of Thm. 1.1. Suppose that there exists a classical solution $\{T, s, z\}$ of (1.1) – (1.6) and consider the related solution $\{T, s, u\}$ of (1.9) – (1.14).

From (1.9) and (1.11) it follows that the assumption $g(x) \leqslant 0$ in $(b - \sigma, b)$ implies that for some $\hat{t} \in (0, T)$ and some $\hat{x} \in (b - \sigma, b)$

(2.1) $\quad u(\hat{x}, t) < 0, \qquad t \in (0, \hat{t}],$

and $s(t) > \hat{x}, \quad t \in (0, \hat{t}]$.

Hence, the strong maximum principle (see e.g. Thms. 4 and 5, Chap. 2 of [6]) applied to the domain $\hat{x} < x < s(t)$, $0 < t \leqslant \hat{t}$ along with Thm. 14, Chap. 2 of [6] leads to the conclusion $u_x(s(t), t) > 0$, $t \in (0, \hat{t}]$, contradicting (1.14). This completes the proof of Thm. 1.1.

(II) Proof of Thm. 1.2. Given a solution $\{T, s, z\}$ to (1.1) – (1.6), for any $t_0 \in (0, T)$ we denote by $\gamma(t_0)$ the spatial coordinate of the right-most point in $D_T \cap \{(x, t) : t = t_0\}$ where $z = -1$; if no such point exists we set $\gamma(t_0) = 0$. In this way a curve $x = \gamma(t)$ is defined and, owing to (H1), $\gamma(0+) \leqslant b - \sigma$. By D^+ we denote the domain $\gamma(t) < x < s(t)$, $0 < t < T$.

Now, let $\{T_i, s_i, z_i\}$, $i = 1, 2$, be two solutions and consider the respective curves $x = \gamma_i(t)$ and domains D_i^+, $i = 1, 2$. From $\gamma_i(0) \leqslant b - \sigma$ it follows that a $\bar{t} > 0$ can be found such that $\hat{D} = D_1^+ \cap D_2^+ \cap \{(x, t) : t < \bar{t}\}$ is nonvoid, connected and has a parabolic boundary.

Passing to the associated solutions $\{T_i, s_i, u_i\}$ to (1.9) – (1.14), from (1.8) we find $u_i > 0$ and $u_{i,x} > 0$ in D_i^+. Therefore in $\hat{D}$ we can consider the difference $u_1 - u_2$ and apply the argument displayed in the proof of Thm. 3.1 of [4] to show that the two solutions coincide.

3 A comparison lemma

In the following lemma $\{T_i, s_i, z_i\}$, $i = 1, 2$ denote two solutions of problem (1.1) – (1.6) with respective data $b_i > 0$, $h_i(x)$, $f_i(t)$. The functions $g_i(x)$ and $u_i(x, t)$ are defined via (1.7) and (1.8) respectively.

Lemma 3.1 *If for some constants t_0, x', x'' satisfying*

$$0 < t_0 \leqslant \min(T_1, T_2), \qquad 0 \leqslant x' \leqslant x'' < \min(b_1, b_2)$$

the conditions (3.1) *through* (3.6) *are valid*

(3.1) $\quad b_1 < b_2,$

(3.2) $\quad g_1(x) \leqslant g_2(x), \qquad x' \leqslant x \leqslant b_1,$

(3.3) $g_2(x) \geqslant 0, \qquad x'' \leqslant x \leqslant b_2,$

(3.4) $s_1(t) > x'', \qquad 0 \leqslant t \leqslant t_0,$

(3.5) $u_2(x'',t) \geqslant 0, \qquad 0 \leqslant t \leqslant \tilde{t} = \sup\{t : 0 < t < t_0, s_2(t) > x''\},$

(3.6) $u_1(x',t) \leqslant u_2(x',t), \qquad 0 \leqslant t \leqslant \tilde{t},$

then

(3.7) $s_1(t) < s_2(t), \qquad 0 \leqslant t \leqslant t_0$

and $\tilde{t} = t_0$.

Proof. Let us suppose that for some t^*, $0 < t^* \leqslant \tilde{t}$, we have $s_1(t^*) = s_2(t^*)$ and $s_1(t) < s_2(t)$ for $0 \leqslant t < t^*$.

Then the difference $w(x,t) = u_2(x,t) - u_1(x,t)$ is such that

$$w_{xx} - w_t = 0, \qquad x' < x < s_1(t), \quad 0 < t \leqslant t^*,$$

$$w(x,0) \geqslant 0, \qquad x' < x < b_1, \quad \text{from (3.1), (3.2),}$$

$$w(x',t) \geqslant 0, \qquad 0 < t \leqslant t^*, \quad \text{from (3.6),}$$

moreover

$$w(s_1(t),t) > 0, \qquad 0 < t < t^*, \quad w(s_1(t^*),t^*) = 0,$$

since $u_2(s_1(t),t) > 0$ for $0 \leqslant t < t^*$ (recall that $u_2 \not\equiv 0$, because $g_2 \equiv 0$ would imply nonexistence of $\{T_2, s_2, z_2\}$: see Thm. 1.1).

Therefore $w(x,t)$ attains its minimum at $x = s_1(t^*)$, $t = t^*$. However, this contradicts the obvious condition $w_x(s_1(t^*), t^*) = 0$ (recall (1.14) and the already quoted theorems of [6]). Thus, $s_2(t) > s_1(t)$ in $[0,\tilde{t}]$, and (3.4) yields $\tilde{t} = t_0$, concluding the proof of the lemma.

Remark 3.1. In the particular case $x' = x'' = 0$, conditions (3.2)–(3.6) are implied by

$$h_1(x) \leqslant h_2(x), \qquad f_1(t) \leqslant f_2(t), \quad \text{and} \quad h_2(x), f_2(t) \geqslant -1,$$

and Lemma 2.1 provides an a priori monotone dependence criterium.

Remark 3.2. Lemma 3.1 remains valid if (3.1) is replaced by $b_1 = b_2$, provided that the inequality $s_1(t) < s_2(t)$ is known to be true in some neighborhood of $t = 0$ (e.g. if $\dot{s}_1(0) < \dot{s}_2(0)$, when such derivatives exist).

4 Existence under assumption (H3)

Assuming (H3), two constants $k < 1$ and $x_0 \in (0,b)$ can be chosen such that

(4.1) $h(x) > -k, \qquad x_0 \leqslant x \leqslant b.$

Consider the function $Z^-(x,t)$ solving

$$
(4.2)\quad \begin{cases} Z^-_{xx} - Z^-_t = 0\,, & x > 0\,, \quad t > 0\,, \\ Z^-(x,0) = \begin{cases} \min(0, h(x))\,, & 0 < x < b\,, \\ 0\,, & b \geqslant x\,, \end{cases} \\ Z^-(0,t) = \min(0, f(t))\,, & 0 < t\,. \end{cases}
$$

Owing to (4.1), $T' > 0$ can be found such that

(4.3) $\quad Z^-(x_0,t) \geqslant -k\,, \qquad 0 < t \leqslant T'.$

For any constant c consider the functions

(4.4) $\quad S(t;c) = c - At^{1/2}\,,$

(4.5) $\quad Z(x,t;c) = K\{1 + \operatorname{erf}[(x-c)/(2t^{1/2})]/\operatorname{erf}(A/2)\}\,,$

where A is the unique solution of the equation

(4.6) $\quad \pi^{1/2}(A/2)\exp(A^2/4)[1 - \operatorname{erf}(A/2)] = k$

and

(4.7) $\quad K = -k/[1 - 1/\operatorname{erf}(A/2)]\,.$

The pair S, Z solves the Stefan problem

$$
(4.8)\quad \begin{cases} Z_{xx} - Z_t = 0\,, & x < S(t)\,, \quad 0 < t\,, \\ S(0;c) = c\,, & \\ Z(x, 0+; c) = -k\,, & x < c\,, \\ Z(S(t), t; c) = 0\,, & 0 < t\,, \\ Z_x(S(t), t; c) = -\dot{S}(t;c)\,, & 0 < t\,. \end{cases}
$$

Fix $a_0 \in (x_0, b)$ and define

(4.9) $\quad T_0 = \sup\{t : 0 < t < T', \;\; Z(x_0, t; a_0) < Z^-(x_0,t)\}.$

For any $c \geqslant a_0$ we have

(4.10) $\quad Z(x_0,t;c) \leqslant Z^-(x_0,t)\,, \qquad 0 \leqslant t \leqslant T_0\,.$

Now we construct a sequence of approximating solutions. We consider two sequences $\{a_n\}$, $\{b_n\} \subset (a_0, b)$, tending monotonically to b and such that

(4.11) $\quad a_n < b_n\,, \qquad a_{n+1} < b_n\,, \qquad b_n - b_{n-1} \geqslant k(a_n - a_{n-1})\,, \qquad n = 1, 2, \ldots$

(such sequences can be constructed very easily). Finally, we define

$$
(4.12)\quad h_n(x) = \begin{cases} h(x)\,, & 0 \leqslant x \leqslant a_n\,, \\ 0\,, & x > a_n\,. \end{cases}
$$

It is known that every problem (1.1) – (1.6) with data $s_n(0) = b_n$, $z_n(x,0) = h_n(x)$, $z_n(0,t) = f(t)$ has a unique classical solution $\{T_n, s_n, z_n\}$ (see [1], Part I).

We shall derive several properties of these approximating solutions to prove their convergence to a solution of (1.1) – (1.6).

Property 1 *For any* $n = 1,2,\ldots$ *and* T_0 *as defined in* (4.9)

(4.13) $T_n \geqslant T_0$

holds true.

Proof. From the maximum principle we get

(4.14) $z_n(x,t) > Z^-(x,t)\,, \qquad 0 < x < s_n(t)\,, \quad 0 < t < T_n\,.$

Hence, recalling (4.10) we have

(4.15) $z_n(x_0,t) > Z(x_0,t;a_0)\,, \qquad 0 < t < \min(T_n, T_0)\,.$

Owing to (4.3) and (4.15) we can use Lemma 3.1 with $x' = x'' = x_0$ to compare $s_n(t)$ and $S(t;a_0)$:

(4.16) $s_n(t) > S(t;a_0) \geqslant x_0\,, \qquad 0 \leqslant t \leqslant \min(T_n, T_0)\,.$

Thus we can conclude that for any n

(4.17) $z_n(x,t) > -k\,, \qquad x_0 < x < s_n(t)\,, \quad 0 < t \leqslant \min(T_n, T_0)\,.$

Recalling Lemma 2.4 of [4] and Thm. 8 of [1], Part I, the inequality $T_n \geqslant T_0$ follows.

Property 2 *For any* $n = 1,2,\ldots$

(4.18) $s_{n+1}(t) > s_n(t)\,, \qquad 0 \leqslant t \leqslant T_0\,.$

Proof. This inequality is an immediate consequence of Lemma 3.1 with $x' = 0$, $x'' = x_0$: actually, using (4.11) and (4.12) one finds

$$\int_x^{b_n} \mathrm{d}\xi \int_\xi^{b_n} [h_n(y) + 1]\,\mathrm{d}y \leqslant \int_x^{b_{n+1}} \mathrm{d}\xi \int_\xi^{b_{n+1}} [h_{n+1}(y) + 1]\,\mathrm{d}y\,,$$

$$n = 1,2,\ldots, \quad 0 < x < b_n\,.$$

Property 3 *For any* $n = 1,2,\ldots$

(4.19) $s_n(t) > S(t;c)\,, \qquad 0 \leqslant t \leqslant T_0\,, \quad$ *for all* $c < b_n\,,$

(4.20) $z_n(x_0,t) > Z(x_0,t;c)\,, \qquad 0 < t < T_0\,, \quad$ *for all* $c \geqslant a_0\,.$

Proof. The first inequality is again a consequence of Lemma 3.1, when $c \geqslant a_0$ and of (4.4) when $c < a_0$; (4.20) follows from (4.10), (4.13) and (4.14).

Property 4 *Suppose for some* n *and some* $\bar{t} \in (0, T_0)$ *there is a value* c *such that*

(4.21) $s_n(\bar{t}) = S(\bar{t};c)\,,$

and

(4.22) $b_n < c$,

then

(4.23) $\begin{cases} s_n(t) < S(t;c), & 0 \leqslant t < \bar{t}, \\ s_n(t) > S(t;c), & \bar{t} < t \leqslant T_0. \end{cases}$

Proof. From (4.21), (4.22) if follows that there is $t_1 < \bar{t}$ such that

$$s_n(t_1) = S(t_1;c) \quad \text{and} \quad s_n(t) < S(t;c) \quad \text{for } 0 \leqslant t < t_1 .$$

First we remark that the difference $s_n(t) - S(t;c)$ must change sign across $t = t_1$: this follows from (4.20), yielding

(4.24) $z_n(x, t_1) > Z(x, t_1;c), \qquad x_0 < x < s_n(t_1)$

and from standard arguments based on the strong maximum principle.

Therefore starting from $t = t_1$ we can apply Lemma 3.1 (modified in the sense of Remark 3.2) to get $s_n(t) > S(t;c)$ for $t_1 < t \leqslant T_0$ (take $x' = x'' = x_0$). The conclusion is that no more than one intersection can occur in $(0, T_0)$, i.e. (4.23).

Property 5 *For any* $t^* \in (0, T_0)$ *either* $\lim s_n(t) = S(t;b)$ *for* $0 \leqslant t \leqslant t^*$, *or there is an* n_0 *such that for* $n > n_0$

(4.25) $z_n(x, t^*) \geqslant Z(x, t^*;s_n(t^*) + A t^{*1/2}), \qquad x_0 \leqslant x \leqslant s_n(t^*)$,

(4.26) $z_n(x, t^*) \leqslant Y(x, t^*), \qquad 0 \leqslant x \leqslant s_n(t^*)$,

where Y *is the solution of the heat equation vanishing on* $x = S(t;s_n(t^*) + A t^{*1/2})$ *and assuming the value*

(4.27) $Y_0 = \max(0, \sup\limits_{x \in (0,b)} h(x), \sup\limits_{t \in (0,T_0)} f(t))$

on $x = 0$ *and on* $t = 0$.

Proof. Fix $t^* \in (0, T_0)$ and recall (4.18). Setting $c = a_n$ in (4.19) and letting $n \to \infty$, we get

(4.28) $\lim s_n(t^*) \geqslant S(t^*;b)$.

If the equality sign holds, then $\lim s_n(t)$ must coincide with $S(t;b)$ for all $t \in [0, t^*]$, otherwise a contradiction to Property 4 could be found, since $s_n(0) < S(0,b)$.

On the other hand, if $\lim s_n(t^*) > S(t^*;b)$, for n greater than some n_0 $s_n(t^*) > S(t^*;b)$, implying $s_n(t^*) + A t^{*1/2} > b > b_n$. Consequently, the curve $x = S(t;s_n(t^*) + A t^{*1/2})$ has its (unique) intersection with $x = s_n(t)$ at $t = t^*$ (see again Property 4). At this point (4.25), (4.26) are easy consequences of the maximum principle.

We are now in position to prove the following existence theorem.

Theorem 4.1 *Under* (H3) *problem* (1.1) – (1.6) *has a (unique) classical solution.*

Proof. Let Y_0 be defined by (4.27). Consider the pair (S_0, Z_0) solving the Stefan problem for the heat equation with data $S_0(0) = b$, $Z_0(0,t) = Y_0, Z_0(x,0) = Y_0$. It is immediately seen that

(4.29) $s_n(t) \leqslant S_0(t)\,, \qquad 0 \leqslant t \leqslant T_0, \quad n = 1,2,\ldots .$

In view of (4.29) and from Property 2 the limit

(4.30) $s(t) = \lim s_n(t)\,, \qquad 0 \leqslant t \leqslant T_0$

exists. On the other hand, Property 5 yields the estimate

(4.31) $|\dot{s}_n(t)| \leqslant Ct^{-1/2}, \qquad 0 < t \leqslant T_0,$

with C depending basically on k and on Y_0. Thus the limit function $s(t)$ is (non-uniformly) Lipschitz continuous in $(0, T_0]$ and the existence proof is concluded by standard arguments (see e.g. [7]).

5 Model problems

In order to complete the proof of Thm. 1.3 we shall need some comparison solutions playing a role similar to the functions S, Z used in the preceding section.

Let $x^* \in (0,b)$, $B > 0$, $r > 1$ be given constants such that $B(b - x^*)^r < 1$. For any $c > b$ define

(5.1) $$h_c(x) = \begin{cases} -1 + B(b-x)^r\,, & x^* \leqslant x \leqslant b \\ -1 + (x-b)/(c-b)\,, & b < x \leqslant c \end{cases}$$

For $x^* < c \leqslant b$, $h_c(x)$ will be defined according to the first of (5.1). Let us look for global classical solutions to the following free boundary problems

(5.2) $z_{xx}(x,t;c) - z_t(x,t;c) = 0\,, \qquad D^* = \{(x,t): x^* < x < s(t;c), t > 0\}\,,$

(5.3) $s(0;c) = c\,,$

(5.4) $z(x,0;c) = h_c(x)\,, \qquad x^* < x < c\,,$

(5.5) $z_x(x^*,t;c) = 0\,, \qquad t > 0\,,$

(5.6) $z(s(t;c),t;c) = 0\,, \qquad t > 0\,,$

(5.7) $z_x(s(t;c),t;c) = -\dot{s}(t;c)\,, \qquad t > 0\,.$

Local existence for any $c \neq b$ is a consequence of Thm. 4.1. Existence for all $t > 0$ follows from the condition

$$\int_{x^*}^{c} h_c(x)\,\mathrm{d}x + c > 0\,,$$

using the techniques of [4]. Moreover $s \in C^\infty(\mathbf{R}^+)$ (see e.g. [3]).

The following monotone dependence result

(5.8) $c_1 < c_2 \Rightarrow s(t;c_1) < s(t;c_2)\,, \qquad t \geqslant 0$

can be deduced duplicating the proof of Lemma 3.1 with $x' = x'' = x^*$, $\bar{t} = t_0 = +\infty$ and (3.6) replaced by $u_{1,x}(x^*,t) = g_1'(x^*), u_{2,x}(x^*,t) = g_2'(x^*)$.

For the time being the condition $c_1, c_2 \neq b$ is tacitly assumed in (5.8); however this exception will be dropped as a consequence of Lemma 5.2 below.

We shall investigate problem (5.2) – (5.7) in the three cases $c < b$, $c = b$, $c > b$.

Lemma 5.1 *Let $x^* < c < b$. Then*

(i) $\ddot{s}(t;c) > 0$;

(ii) *for any $t' > 0$, $|\dot{s}(t';c)|$ is bounded independently of c.*

Proof. To prove (i), we state the following facts, mainly based on the maximum principle (we omit the arguments of s, when unambiguous).

(a) It is $z_x(s,t;c) > 0, t > 0$ and $\lim_{t \to 0} z_x(s,t;c) = +\infty$.

(b) There exists a $\hat{t} > 0$ such that $z_{xx}(x^*,t;c) < 0$ for $0 < t < \hat{t}$, $z_{xx}(x^*,\hat{t};c) = 0$.

(c) There is a unique curve γ_1 originating from $(c,0)$, where $z_x = 0$, and a unique curve γ_2 originating from $(x^*,0)$, where $z_{xx} = 0$.

(d) γ_1 and γ_2 have their unique intersection in $(x^*,\hat{t})$.

Now, consider the part of D^* (defined in (5.2)) lying above γ_1. In this open set $z_x, z_{xx} > 0$ and $v \equiv z_{xx}/z_x$ is defined and positive. The level curves of v which originate from points of the free boundary are directed upwards and go to infinity. Then (i) follows from the arguments of [8] (or [9], see Lemma 3.2).

Clearly, (i) implies that for any fixed $t' > 0$ the straight line connecting the points $(b,0)$ and $(s(t';c),t')$ lies to the right of $x = s(t;c)$ for $0 < t < t'$. Moreover, its slope decreases as c increases, because of (5.8). This provides a very simple tool to construct barriers for z at $(s(t';c),t')$. Hence (ii) is proved.

As an immediate consequence of Lemma 5.1 we have the following existence result for a model problem whose initial datum has the value -1 at $x = b$.

Lemma 5.2 *Problem (5.2) – (5.7) with $c = b$ has a unique classical solution existing globally. Moreover $\ddot{s}(t;b) > 0$.*

Proof. Uniqueness follows from Theorem 1.2. Existence is an easy consequence of (5.8) and (ii) of Lemma 5.1 (recall also the trivial estimate $s(t;c) < b$, $c < b$). Convexity can be proved as above.

Next, we have

Lemma 5.3 *Let $c > b$. Then:*

(i) *for any* $t' > 0$, $|\dot{s}(t';c)|$ *is bounded independently of* c;

(ii) *for any point* $(\bar{x}, \bar{t})$, $\bar{t} > 0$, $\bar{x} > s(\bar{t}, b)$, *there exists a* $\bar{c} > b$ *such that* $s(\bar{t};\bar{c}) = \bar{x}$.

Proof. In this case, the curve γ_1 (see point (c) of the proof of Lemma 5.1) originates from $(b, 0)$ instead of $(c, 0)$. Thus, the arguments of [9] can be adapted (see in particular Theorem 3.3, (iv)) to conclude that a $t_c > 0$ exists such that

$$\ddot{s}(t;c) < 0, \qquad 0 < t < t_c,$$

$$\ddot{s}(t;c) > 0, \qquad t_c < t.$$

On the other hand, $\dot{s}(0,c) = (b - c)^{-1}$, tends to $-\infty$ as c tends to b, yielding

$$\lim_{c \to b+} t_c = 0. \tag{5.9}$$

From (5.9), once $t' > 0$ is fixed we can find c_0 such that

$$t_c < t'/2, \quad \text{for } c \in (b, c_0).$$

Therefore, a barrier for z at $(s(t',c), t')$ can be constructed by solving the heat equation in a domain bounded on its right hand side by a polygonal $x = \lambda_c(t)$ defined by:

$$\lambda_c(t) = \begin{cases} c, & 0 \leqslant t < t'/2, \\ c + 2(t - t'/2) \cdot [s(t',c) - c]/t', & t'/2 \leqslant t \leqslant t'. \end{cases}$$

Since $\dot{\lambda}_c > 2\{s(t',b) - c_0\}/t'$, a uniform bound on $\dot{s}(t';c)$ is obtained independently of $c < c_0$. When $c \geqslant c_0$ the same result can be proved as in [1,I]. This concludes the proof of (i). By the way, we note that $s(t;c)$ tends to $s(t;b)$ uniformly in any compact time interval not only as $c \to b^-$, as proved in Lemma 5.2, but also for $c \to b^+$.

To prove (ii), we remark that from (5.2)–(5.7) the following integral relationship is easily found to hold:

$$s(\bar{t};c) = (3c - b)/2 - \int_{x^*}^{b} [1 - B(b - x)^r]\,dx - \int_{x^*}^{s(\bar{t};c)} z(x, \bar{t};c)\,dx.$$

Hence, letting $c \to +\infty$, s cannot remain bounded, since z is bounded. Moreover, $s(t;c)$ is seen to depend continuously on c for $c > b$: this ensues from Thm. 5 of [1], Part I, and from the convergence of $s(t;c)$ to $s(t;b)$, recalled above.

Remark 5.1. The results of this section remain true if the function $h_c(x)$ coincides in the interval $x^* \leqslant x \leqslant b$ with a function $\bar{h}(x) \in C^2[x^*, b]$ such that $\bar{h}(b) = -1$, $\bar{h}'(b) = 0$, $\bar{h}'' > 0$ for $x^* < x < b$, and $\bar{h}(x^*) < 0$.

6 Proof of Theorem 1.3

Under assumption (H2) it is possible to find $\tilde{h}$ as in Rem. 5.1 in such a way that

$$(6.1)\quad h(x) \geqslant \tilde{h}(x)\,, \qquad x^* \leqslant x \leqslant b\,.$$

Let $\{b_n\} \subset (x^*, b)$ be a sequence converging monotonically to b and consider the solutions $\{\tilde{T}_n, \tilde{s}_n, \tilde{z}_n\}$ of problem (1.1) – (1.6) with b replaced by b_n in (1.2) and (1.3). Their existence and uniqueness is ensured by (6.1) and Thm. 4.1.

Recalling the definitions of $Z^-(x,t)$ (see (4.2)) and of $z(x,t;b_n)$ (see Sec. 5), we define

$$(6.2)\quad T' = \sup\{t : Z^-(x^*,t) > z(x^*,t;b_1)\}\,.$$

The inequalities

$$(6.3)\quad -1 < z(x^*,t;c) < Z^-(x^*,t) < \tilde{z}_n(x^*,t)\,, \qquad 0 < t < \min(T',\tilde{T}_n)$$

are easily verified for any $c > b_1$. Thus, Lemma 3.1 can be applied ($x' = x'' = x^*$) to conclude that, for any $c \in (b_1, b_n)$,

$$(6.4)\quad s(t;c) < \tilde{s}_n(t)\,, \qquad 0 < t < \min(T',\tilde{T}_n)\,,$$

whence the uniform estimate

$$(6.5)\quad \tilde{T}_n \geqslant T'\,, \qquad n = 1,2,\ldots,$$

follows from the same arguments used in proving Property 1, Sec. 4.

As a consequence of Lemma 3.1:

$$(6.6)\quad S_0(t) > \tilde{s}_{n+1}(t) > \tilde{s}_n(t)\,, \qquad 0 \leqslant t \leqslant T'\,, \quad n = 1,2,\ldots,$$

where $S_0(t)$ is the function appearing in (4.28).

A result similar to Property 4, Sec. 4 can be proved in the same way, using (6.3): if for some $\bar{t} \in (0,T')$ and some $c > b_n$

$$(6.7)\quad \tilde{s}_n(\bar{t}) = s(\bar{t};c)\,,$$

then

$$(6.8)\quad \begin{cases} \tilde{s}_n(t) < s(t;c)\,, & 0 \leqslant t < \bar{t}\,, \\ \tilde{s}_n(t) > s(t;c)\,, & \bar{t} < t \leqslant T'\,. \end{cases}$$

Finally, fix $t^* \in (0,T')$. From (6.4) we deduce that $\tilde{s}(t) \equiv \lim \tilde{s}_n(t) \geqslant s(t;b)$; hence, if $\tilde{s}(t^*) = s(t^*;b)$, then $\tilde{s}(t) = s(t;b)$ for $0 \leqslant t \leqslant t^*$ (since otherwise the implication (6.7) $\Rightarrow$ (6.8) would be violated for $c = b$ and n large enough).

It remains to consider the case $\tilde{s}(t^*) > s(t^*;b)$, i.e. $\tilde{s}_n(t^*) > s(t^*,b)$ for $n > n_0$, large enough. Fix $n > n_0$ and consider the curve $x = s(t;c^*)$ such that $s(t^*;c^*) = \tilde{s}_n(t^*)$ (see Lemma 5.3., (ii)).

Since such curve lies to the right of $x = \tilde{s}_n(t)$, upper and lower bounds for $\dot{\tilde{s}}_n(t^*)$ (independent of n and possibly diverging for $t^* \to 0^+$) are easily obtained recalling Lemma 5.3 (i), which permits the construction of simple barriers (as we did in proving (4.26)). At this point, the last step of the proof is standard.

References

[1] Fasano, A.; Primicerio, M.: General free boundary problems for the heat equation, Part I. J. Math. Anal. Appl. **57** (1977) 694–723; Part II, ibid. **58** (1977) 202–231; Part III, ibid. **59** (1977) 1–14

[2] Cannon, J. R.; Hill, C. D.; Primicerio, M.: The one-phase Stefan problem for the heat equation with boundary temperature specification. Arch. Rat. Mech. Anal. **39** (1970) 270–274

[3] Rubinstein, L.; Fasano, A.; Primicerio, M.: Remarks on the analyticity of the free boundary for the one-dimensional Stefan problem. Ann. Mat. Pura Appl. (IV) **125** (1980) 295–311

[4] Fasano, A.; Primicerio, M.: New results on some classical parabolic free boundary problem. Quart. Appl. Math. **38** (1981) 439–460

[5] Baiocchi, C.; Pozzi, G.: An evolution variational inequality related to a diffusion-absorption problem. Appl. Math. Optim. **2** (1976) 304–314

[6] Friedman, A.: Partial differential equations of parabolic type. Englewood Cliffs: Prentice Hall 1964

[7] Cannon, J. R.; Hill, C. D.: Existence, uniqueness, stability and monotone dependence in a Stefan problem for the heat equation. J. Math. Mech. **17** (1967) 1–10

[8] Friedman, A.; Jensen, R.: Convexity of the free boundary in the Stefan problem and in the dam problem. Arch. Rat. Mech. Anal. **67** (1977) 1–24

[9] Fasano, A.; Primicerio, M.: Convexity of the free boundary in some classical parabolic free boundary problems. Riv. Mat. Univ. Parma (4) **5** (1979) 635–645

Antonio Fasano
Mario Primicerio
Istituto Matematico "U. Dini"
viale Morgagni 67a
I 50134 Firenze, Italy

*(Received April 3, 1981,
Revised October 5, 1981)*